

# QUANTUM MEASUREMENT, INFORMATION, AND COMPLETELY POSITIVE MAPS

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## Abstract

Axiomatic approach to measurement theory is developed. All the possible statistical properties of apparatuses measuring an observable with nondegenerate spectrum allowed in standard quantum mechanics are characterized.

## 1. INTRODUCTION

Every measuring apparatus inputs the state  $\rho$  of the measured system and outputs the classical output  $\mathbf{x}$  and the state  $\rho_{\{\mathbf{x}=x\}}$  of the measured system conditional upon the outcome  $\mathbf{x} = x$ . In the conventional approach, the probability distribution of the classical output  $\mathbf{x}$  and the output state  $\rho_{\{\mathbf{x}=x\}}$  are determined by the spectral projections of the measured observable by the Born statistical formula and the projection postulate, respectively. This description has been a very familiar principle in quantum mechanics, but is much more restrictive than what quantum mechanics allows. In the modern measurement theory, the problem has been investigated as to what is the most general description of measurement allowed by quantum mechanics. This paper investigates the problem of the determination of all the possible measurements of observables with nondegenerate spectrum and shows that the following conditions are equivalent for measurements of nondegenerate observables: (i) The joint probability distribution of the outcomes of successive measurements depends affinely on the initial state. (ii) The apparatus has an indirect measurement model. (iii) The state change is described by a positive superoperator valued measure. (iv) The state change is described by a completely positive superoperator valued measure. (v) The family of output states is a Borel family of density operators independent of the input state and can be arbitrarily chosen by the choice of the apparatus.

## 2. MEASUREMENT SCHEMES

Let  $\mathcal{H}$  be a separable Hilbert space. The *state space* of  $\mathcal{H}$  is the set  $\mathcal{S}(\mathcal{H})$  of density operators on  $\mathcal{H}$ . In what follows, we shall give a general mathematical formulation for the statistical properties of measuring apparatuses. For heuristics, we shall consider a measuring apparatus which measures the quantum system  $\mathbf{S}$  described by the Hilbert space  $\mathcal{H}$ . Every measuring apparatus has the output variable that gives the outcome on each measurement. We assume that the output variable takes values in a standard Borel space which is specified by each measuring apparatus. We shall

denote by  $\mathbf{A}(\mathbf{x})$  the measuring apparatus with the output variable  $\mathbf{x}$  taking values in a standard Borel space  $\Lambda$  with the Borel  $\sigma$ -field  $\mathcal{B}(\Lambda)$ . The statistical property of the apparatus  $\mathbf{A}(\mathbf{x})$  consists of the output distribution  $\Pr\{\mathbf{x} \in \Delta \|\rho\}$  and the state reduction  $\rho \mapsto \rho_{\{\mathbf{x}=x\}}$ . The output distribution  $\Pr\{\mathbf{x} \in \Delta \|\rho\}$  describes the probability distribution of the output variable  $\mathbf{x}$  when the input state is  $\rho \in \mathcal{S}(\mathcal{H})$ , where  $\Delta \in \mathcal{B}(\Lambda)$ . The state reduction  $\rho \mapsto \rho_{\{bx=x\}}$  describes the state change from the input state  $\rho$  to the output state  $\rho_{\{\mathbf{x}=x\}}$ , when the measurement leads to the output  $\mathbf{x} = x$ . The output state  $\rho_{\{\mathbf{x}=x\}}$  is determined up to probability one with respect to the output distribution. The state reduction determines the collective state reduction  $\rho \mapsto \rho_{\{\mathbf{x} \in \Delta\}}$  that describes the output state  $\rho_{\{\mathbf{x} \in \Delta\}}$  given that the output of the measurement is in a Borel set  $\Delta$ . The collective state reduction is naturally related to the state reduction by the integral formula

$$\rho_{\{\mathbf{x} \in \Delta\}} = \frac{1}{\Pr\{\mathbf{x} \in \Delta \|\rho\}} \int_{\Delta} \rho_{\{\mathbf{x}=x\}} \Pr\{\mathbf{x} \in dx \|\rho\}.$$

Formal description of the statistical properties of measuring apparatuses will be given as follows.

Let  $\Lambda$  be a standard Borel space with  $\sigma$ -field  $\mathcal{B}(\Lambda)$ . Denote by  $B(\Lambda, \mathcal{S}(\mathcal{H}))$  the space of Borel families of states for  $(\mathcal{H}, \Lambda)$ . The *state space* of  $\Lambda$  is the set  $\mathcal{S}(\Lambda)$  of probability measures on  $\mathcal{B}(\Lambda)$ . A *measurement scheme* for  $(\mathcal{H}, \Lambda)$  is the pair  $(\mathcal{P}, \mathcal{Q})$  of a function  $\mathcal{P}$  from  $\mathcal{S}(\mathcal{H})$  to  $\mathcal{S}(\Lambda)$  and a function  $\mathcal{Q}$  from  $\mathcal{S}(\mathcal{H})$  to  $B(\Lambda, \mathcal{S}(\mathcal{H}))$ . The function  $\mathcal{P}$  is called the *output probability scheme*, and  $\mathcal{Q}$  is called the *state reduction scheme*. Two measurement schemes  $(\mathcal{P}, \mathcal{Q})$  and  $(\mathcal{P}', \mathcal{Q}')$  are said to be *equivalent*, in symbols  $(\mathcal{P}, \mathcal{Q}) \cong (\mathcal{P}', \mathcal{Q}')$ , if  $\mathcal{P} = \mathcal{P}'$  and  $\mathcal{Q}\rho$  and  $\mathcal{Q}'\rho$  differ only on a null set of the probability measure  $\mathcal{P}\rho$ , i.e.,

$$(\mathcal{P}\rho)\{x \in \Lambda \mid (\mathcal{Q}\rho)(x) \neq (\mathcal{Q}'\rho)(x)\} = 0,$$

for all  $\rho \in \mathcal{S}(\mathcal{H})$ .

The set of equivalence classes of measurement schemes for  $(\mathcal{H}, \Lambda)$  is denoted by  $\mathcal{M}(\mathcal{H}, \Lambda)$  and we define  $\mathcal{M}(\mathcal{H}) = \bigcup_{\Lambda} \mathcal{M}(\mathcal{H}, \Lambda)$ . A *measurement theory* for  $\mathcal{H}$  is a pair  $(\mathcal{A}, \mathbf{M})$  consisting of a nonempty set  $\mathcal{A}$  and a function  $\mathbf{M}$  from  $\mathcal{A}$  to  $\mathcal{M}(\mathcal{H})$ . An element of  $\mathcal{A}$  is called an *apparatus*. Every apparatus has its distinctive *output variable*. We denote the apparatuses with output variables  $\mathbf{x}, \mathbf{y}, \dots$  by  $\mathbf{A}(\mathbf{x}), \mathbf{A}(\mathbf{y}), \dots$ , respectively. We assume that  $\mathbf{x} = \mathbf{y}$  if and only if  $\mathbf{A}(\mathbf{x}) = \mathbf{A}(\mathbf{y})$ . The image of  $\mathbf{A}(\mathbf{x})$  by  $\mathbf{M}$  is denoted by  $\mathbf{M}(\mathbf{x})$  instead of  $\mathbf{M}(\mathbf{A}(\mathbf{x}))$  for simplicity, so that  $\mathbf{M}(\mathbf{x})$  denotes the equivalence class of the measurement scheme corresponding to the apparatus  $\mathbf{A}(\mathbf{x})$ . The apparatus  $\mathbf{A}(\mathbf{x})$  is called  *$\Lambda$ -valued* if  $\mathbf{M}(\mathbf{x}) \in \mathcal{M}(\mathcal{H}, \Lambda)$ . For any  $\mathbf{A}(\mathbf{x}) \in \mathcal{A}$ , the equivalence class  $\mathbf{M}(\mathbf{x})$  of the measurement scheme is called the *statistical property of  $\mathbf{A}(\mathbf{x})$* . We shall denote by  $(\mathcal{P}_{\mathbf{x}}, \mathcal{Q}_{\mathbf{x}})$  a representative of  $\mathbf{M}(\mathbf{x})$ ; in this case, we shall also write  $\mathbf{M}(\mathbf{x}) = [\mathcal{P}_{\mathbf{x}}, \mathcal{Q}_{\mathbf{x}}]$ . The function  $\mathcal{P}_{\mathbf{x}}$  is called the *output probability* of  $\mathbf{A}(\mathbf{x})$  and  $\mathcal{Q}_{\mathbf{x}}$  the *state reduction* of  $\mathbf{A}(\mathbf{x})$  which is determined uniquely up to the output probability one. Two apparatuses  $\mathbf{A}(\mathbf{x})$  and  $\mathbf{A}(\mathbf{y})$  are said to be *statistically equivalent*, in symbols  $\mathbf{A}(\mathbf{x}) \cong \mathbf{A}(\mathbf{y})$ , if they have the same statistical property, i.e.,  $\mathbf{M}(\mathbf{x}) = \mathbf{M}(\mathbf{y})$ .

Suppose that a measurement theory  $(\mathcal{A}, \mathbf{M})$  is given. Let  $\mathbf{A}(\mathbf{x}) \in \mathcal{A}$  and  $\mathbf{M}(\mathbf{x}) = [\mathcal{P}_{\mathbf{x}}, \mathcal{Q}_{\mathbf{x}}]$ . The *probability distribution* of the output variable  $\mathbf{x}$  in the state  $\rho$  is defined by

$$\Pr\{\mathbf{x} \in \Delta \|\rho\} = (\mathcal{P}_{\mathbf{x}}\rho)(\Delta)$$

for all  $\rho \in \mathcal{S}(\mathcal{H})$  and  $\Delta \in \mathcal{B}(\Lambda)$ . This probability distribution is called the *output distribution* of  $\mathbf{A}(\mathbf{x})$  in  $\rho$ . The *output states*  $\{\rho_{\{\mathbf{x}=x\}}\}_{x \in \Lambda}$  of  $\mathbf{A}(\mathbf{x})$  in  $\rho$  is defined by

$$\rho_{\{\mathbf{x}=x\}} = (\mathcal{Q}_{\mathbf{x}}\rho)(x)$$

for all  $\rho \in \mathcal{S}(\mathcal{H})$  and  $x \in \Lambda$ .

Let  $\Lambda_1, \dots, \Lambda_n$  be standard Borel spaces. For  $j = 1, \dots, n$ , let  $\mathbf{A}(\mathbf{x}_j)$  be a  $\Lambda_j$ -valued apparatus. A *successive measurement in the input state*  $\rho$  is a sequence of measurements using  $\mathbf{A}(\mathbf{x}_1), \dots, \mathbf{A}(\mathbf{x}_n)$  such that the input state of the apparatus  $\mathbf{A}(\mathbf{x}_1)$  is  $\rho$  and the input state of the apparatus  $\mathbf{A}(\mathbf{x}_{j+1})$  is the output state of the apparatus  $\mathbf{A}(\mathbf{x}_j)$  for  $j = 1, \dots, n-1$ . The joint probability distribution of the outcomes of the successive measurements using  $\mathbf{A}(\mathbf{x}_1), \dots, \mathbf{A}(\mathbf{x}_n)$  in the input state  $\rho$  is naturally defined recursively by

$$\begin{aligned} & \Pr\{\mathbf{x}_1 \in \Delta_1, \dots, \mathbf{x}_n \in \Delta_n \|\rho\} \\ &= \int_{\Delta_1} \Pr\{\mathbf{x}_2 \in \Delta_2, \dots, \mathbf{x}_n \in \Delta_n \|\rho_{\{\mathbf{x}_1=x_1\}}\} \Pr\{\mathbf{x}_1 \in dx_1 \|\rho\} \end{aligned} \quad (1)$$

for  $\Delta_1 \in \mathcal{B}(\Lambda_1), \dots, \Delta_n \in \mathcal{B}(\Lambda_n)$ .

Now, we consider the following requirement for a measurement theory  $(\mathcal{A}, \mathbf{M})$ :

**Mixing law of the  $n$ -joint probability distributions (nMLPD):** For any sequence  $\mathbf{A}(\mathbf{x}_1), \dots, \mathbf{A}(\mathbf{x}_n)$  of apparatuses with values in  $\Lambda_1, \dots, \Lambda_n$ , respectively, if the input state  $\rho$  is the mixture of  $\rho_1$  and  $\rho_2$  such that  $\rho = \alpha\rho_1 + (1-\alpha)\rho_2$  with  $0 < \alpha < 1$  then we have

$$\begin{aligned} & \Pr\{\mathbf{x}_1 \in \Delta_1, \dots, \mathbf{x}_n \in \Delta_n \|\rho\} \\ &= \alpha \Pr\{\mathbf{x}_1 \in \Delta_1, \dots, \mathbf{x}_n \in \Delta_n \|\rho_1\} \\ &+ (1-\alpha) \Pr\{\mathbf{x}_1 \in \Delta_1, \dots, \mathbf{x}_n \in \Delta_n \|\rho_2\} \end{aligned} \quad (2)$$

for all  $\rho \in \mathcal{S}(\mathcal{H})$  and  $\Delta_1 \in \mathcal{B}(\Lambda_1), \dots, \Delta_n \in \mathcal{B}(\Lambda_n)$ .

An *observable* of  $\mathcal{H}$  is a self-adjoint operator (densely defined) on  $\mathcal{H}$ . We denote by  $E^A$  the spectral measure of an observable  $A$ . According to the Born statistical formula, we say that an  $\mathbf{R}$ -valued apparatus  $\mathbf{A}(\mathbf{x})$  *measures an observable*  $A$  if

$$\Pr\{\mathbf{x} \in \Delta \|\rho\} = \text{Tr}[E^A(\Delta)\rho] \quad (3)$$

for all  $\rho \in \mathcal{S}(\mathcal{H})$  and  $\Delta \in \mathcal{B}(\mathbf{R})$ , where  $\mathbf{R}$  stands for the real number field. A measurement theory  $(\mathcal{A}, \mathbf{M})$  is called *nonsuperselective* if for any observable  $A$  there is at least one apparatus measuring  $A$ .

A  $\Lambda$ -valued *observable* of  $\mathcal{H}$  is a projection valued measure  $E$  from  $\mathcal{B}(\Lambda)$  to  $\mathcal{L}(\mathcal{H})$  such that  $E(\Lambda) = I$ . The Born statistical formula is generalized as follows. We say that a  $\Lambda$ -valued apparatus  $\mathbf{A}(\mathbf{x})$  *measures a  $\Lambda$ -valued observable*  $E$  if

$$\Pr\{\mathbf{x} \in \Delta \|\rho\} = \text{Tr}[E(\Delta)\rho] \quad (4)$$

for all  $\rho \in \mathcal{S}(\mathcal{H})$  and  $\Delta \in \mathcal{B}(\Lambda)$ . A *probability operator valued measure (POVM)* for  $(\mathcal{H}, \Lambda)$  is a positive operator valued measure  $F$  from  $\mathcal{B}(\Lambda)$  to  $\mathcal{L}(\mathcal{H})$  such that  $F(\Lambda) = I$ . We say that a  $\Lambda$ -valued apparatus  $\mathbf{A}(\mathbf{x})$  *measures a POVM*  $F$  for  $(\mathcal{H}, \Lambda)$  if

$$\Pr\{\mathbf{x} \in \Delta \|\rho\} = \text{Tr}[F(\Delta)\rho] \quad (5)$$

for all  $\rho \in \mathcal{S}(\mathcal{H})$  and  $\Delta \in \mathcal{B}(\Lambda)$ . Conventional measurement theory is devoted to measurements of observables but modern theory extends the notion of measurements to measurements of POVMs [1–5]. We shall describe in the following requirement the essential feature of the modern approach.

**Existence of probability operator valued measures (EPOVM):** *For any apparatus  $\mathbf{A}(\mathbf{x})$ , there exists a POVM  $F_{\mathbf{x}}$  uniquely such that  $\mathbf{A}(\mathbf{x})$  measures  $F_{\mathbf{x}}$ .*

The EPOVM is justified by the following theorem proved in [3].

**Theorem 1** *For any measurement theory  $(\mathcal{A}, \mathbf{M})$ , the EPOVM is equivalent to the 1MLPD.*

### 3. COLLECTIVE MEASUREMENT SCHEMES

In order to provide an alternative definition of measurement schemes, we call a pair  $(\mathcal{P}, \mathcal{R})$  as a *collective measurement scheme* if  $\mathcal{P}$  is a function from  $\mathcal{S}(\mathcal{H})$  to  $\mathcal{S}(\Lambda)$  and  $\mathcal{R}$  is a function from  $\mathcal{B}(\Lambda) \times \mathcal{S}(\mathcal{H})$  to  $\mathcal{S}(\mathcal{H})$  satisfying

$$\sum_n (\mathcal{P}\rho)(\Delta_n) \mathcal{R}(\Delta_n, \rho) = \mathcal{R}(\Lambda, \rho) \quad (6)$$

for any countable Borel partition  $\{\Delta_1, \Delta_2, \dots\}$  of  $\Lambda$  and  $\rho \in \mathcal{S}(\mathcal{H})$ , where the sum is convergent in the trace norm. The function  $\mathcal{R}$  is called the *collective reduction scheme*. Two collective measurement schemes  $(\mathcal{P}, \mathcal{R})$  and  $(\mathcal{P}', \mathcal{R}')$  are said to be *equivalent*, in symbols  $(\mathcal{P}, \mathcal{R}) \cong (\mathcal{P}', \mathcal{R}')$ , if  $\mathcal{P} = \mathcal{P}'$  and  $\mathcal{R}(\Delta, \rho) = \mathcal{R}'(\Delta, \rho)$  for all  $\Delta \in \mathcal{B}(\Lambda)$  with  $(\mathcal{P}\rho)(\Delta) > 0$ .

**Theorem 2** *The relation*

$$(\mathcal{P}\rho)(\Delta) \mathcal{R}(\Delta, \rho) = \int_{\Delta} (\mathcal{Q}\rho)(x) d(\mathcal{P}\rho)(x) \quad (7)$$

where  $\Delta \in \mathcal{B}(\Lambda)$  and  $\rho \in \mathcal{S}(\mathcal{H})$ , sets up a one-to-one correspondence between the equivalence classes of measurement schemes  $(\mathcal{P}, \mathcal{Q})$  and the equivalence classes of collective measurement schemes  $(\mathcal{P}, \mathcal{R})$ .

Let  $(\mathcal{P}, \mathcal{Q})$  be a measurement scheme for  $(\mathcal{H}, \Lambda)$ . The collective measurement scheme  $(\mathcal{P}, \mathcal{R})$  defined by (7) up to equivalence is called the *collective measurement scheme induced by  $(\mathcal{P}, \mathcal{Q})$*  and the function  $\mathcal{R}$  is called the *collective reduction scheme induced by  $(\mathcal{P}, \mathcal{Q})$* .

Let  $(\mathcal{A}, \mathbf{M})$  be a measurement theory satisfying the 1MLPD. For any  $\mathbf{A}(\mathbf{x}) \in \mathcal{A}$ , define  $\mathcal{R}_{\mathbf{x}}$  to be the collective reduction scheme induced by  $(\mathcal{P}_{\mathbf{x}}, \mathcal{Q}_{\mathbf{x}})$ . The functions  $\mathcal{R}_{\mathbf{x}}$  is called the *collective reduction* of the apparatus  $\mathbf{A}(\mathbf{x})$ . The *collective output states*  $\{\rho_{\{\mathbf{x} \in \Delta\}}\}_{\Delta \in \mathcal{B}(\Lambda)}$  of  $\mathbf{A}(\mathbf{x})$  in  $\rho$  is defined by

$$\rho_{\{\mathbf{x} \in \Delta\}} = \mathcal{R}_{\mathbf{x}}(\Delta, \rho)$$

for all  $\rho \in \mathcal{S}(\mathcal{H})$  and  $\Delta \in \mathcal{B}(\Lambda)$ .

## 4. DAVIES-LEWIS POSTULATE

In what follows, we shall introduce some mathematical terminology independent of particular measurement theory. A *superoperator* for  $\mathcal{H}$  is a bounded linear transformation on the space  $\tau c(\mathcal{H})$  of trace class operators on  $\mathcal{H}$ . The *dual* of a superoperator  $L$  is the dual superoperator  $L^*$  defined by  $\langle L^* A, \rho \rangle = \langle A, L\rho \rangle$  for all  $A \in \mathcal{L}(\mathcal{H})$  and  $\rho \in \tau c(\mathcal{H})$ , where  $\langle \cdot, \cdot \rangle$  stands for the duality pairing defined by  $\langle A, \rho \rangle = \text{Tr}[A\rho]$  for all  $A \in \mathcal{L}(\mathcal{H})$  and  $\rho \in \tau c(\mathcal{H})$ . A superoperator is called *positive* if it maps positive operators to positive operators. We denote by  $\mathcal{P}(\tau c(\mathcal{H}))$  the space of positive superoperators. Positive contractive superoperators are called *operations* [6].

A *positive superoperator valued (PSV) measure* is a mapping  $\mathcal{I}$  from  $\mathcal{B}(\Lambda)$  to  $\mathcal{P}(\tau c(\mathcal{H}))$  such that if  $\{\Delta_1, \Delta_2, \dots\}$  is a countable Borel partition of  $\Lambda$ , then we have

$$\mathcal{I}(\Lambda)\rho = \sum_n \mathcal{I}(\Delta_n)\rho$$

for any  $\rho \in \tau c(\mathcal{H})$ , where the sum is convergent in the trace norm. The PSV measure  $\mathcal{I}$  is said to be *normalized* if it satisfies the further condition

$$\text{Tr}[\mathcal{I}(\Lambda)\rho] = \text{Tr}[\rho]$$

for any  $\rho \in \tau c(\mathcal{H})$ . Normalized PSV measures are called *instruments* [2,8] for short.

Let  $\mathcal{I}$  be an instrument for  $(\Lambda, \mathcal{H})$ . The relation

$$X(\Delta) = \mathcal{I}(\Delta)^* I \quad (8)$$

for all  $\Delta \in \mathcal{B}(\Lambda)$ , defines a POVM for  $(\mathcal{H}, \Lambda)$ , called the *POVM* of  $\mathcal{I}$ . The relation  $T = \mathcal{I}(\Lambda)$  defines a trace preserving operation, called the *total operation* of  $\mathcal{I}$ .

A measurement theory  $(\mathcal{A}, \mathbf{M})$  is said to satisfy the *Davies-Lewis postulate* if it satisfies the follows postulate.

**Davies-Lewis postulate:** *For any apparatus  $\mathbf{A}(\mathbf{x})$ , there is a normalized PSV measure  $\mathcal{I}_{\mathbf{x}}$  satisfying the following relations for any  $\rho \in \mathcal{S}(\mathcal{H})$  and Borel set  $\Delta \in \mathcal{B}(\Lambda)$ :*

$$(DL1) \text{Pr}\{\mathbf{x} \in \Delta | \rho\} = \text{Tr}[\mathcal{I}_{\mathbf{x}}(\Delta)\rho].$$

$$(DL2) \rho_{\{\mathbf{x} \in \Delta\}} = \frac{\mathcal{I}_{\mathbf{x}}(\Delta)\rho}{\text{Tr}[\mathcal{I}_{\mathbf{x}}(\Delta)\rho]}.$$

From Theorem 2, the normalized PSV measure  $\mathcal{I}_{\mathbf{x}}$  determines the output state  $\rho_{\{\mathbf{x}=x\}}$  uniquely up to equivalence by

$$(DL3) \int_{\Delta} \rho_{\{\mathbf{x}=x\}} \text{Tr}[d\mathcal{I}_{\mathbf{x}}(x)\rho] = \mathcal{I}_{\mathbf{x}}(\Delta)\rho.$$

From (DL1), the Davies-Lewis postulate implies 1MLPD. Although the Davies-Lewis description of measurement is quite general, it is not clear by itself whether it is general enough to allow all the possible measurements. The following theorem shows indeed it is the case.

**Theorem 3** *For any nonsuperselective measurement theory, the Davies-Lewis postulate is equivalent to the 2MLPD.*

A measurement theory  $(\mathcal{A}, \mathbf{M})$  is called a *statistical measurement theory* if it is nonsuperselective and satisfies 2MLPD.

**Theorem 4** *Every statistical measurement theory satisfies nMLPD for all positive integer  $n$ .*

## 5. MEASUREMENTS OF OBSERVABLES

An instrument  $\mathcal{I}$  for  $(\Lambda, \mathcal{H})$  is said to be *decomposable* if  $\mathcal{I}(\Delta)^*A = X(\Delta)T^*(A)$  for all  $\Delta \in \mathcal{B}(\Lambda)$  and  $A \in \mathcal{L}(\mathcal{H})$ , where  $X$  is the POVM of  $\mathcal{I}$  and  $T$  the total operation. For a given  $\Lambda$ -valued observable  $E$  for  $(\Lambda, \mathcal{H})$ , an instrument  $\mathcal{I}$  is said to be *E-compatible* if the POVM of  $\mathcal{I}$  is  $E$ . i.e.,  $\mathcal{I}^*(\Delta)I = E(\Delta)$  for all  $\Delta \in \mathcal{B}(\Lambda)$ ; such an instrument is also called *observable measuring*. For an observable  $A$ , an instrument is called *A-compatible* if it is  $E^A$ -compatible. The following theorem shows in particular that every observable measuring instrument is decomposable (see [5, Proposition 4.3] for the case of completely positive instruments).

**Theorem 5** *Let  $E$  be a  $\Lambda$ -valued observable of  $\mathcal{H}$ . Let  $\mathcal{I}$  be an  $E$ -compatible instrument and  $T$  its total operation. Then, we have the following statements.*

(i) *For any  $\Delta \in \mathcal{B}(\Lambda)$  and  $\rho \in \tau c(\mathcal{H})$ , we have*

$$\mathcal{I}(\Delta)\rho = T[E(\Delta)\rho] = T[\rho E(\Delta)] = T[E(\Delta)\rho E(\Delta)]. \quad (9)$$

(ii) *For any  $\Delta \in \mathcal{B}(\Lambda)$  and  $B \in \mathcal{L}(\mathcal{H})$ , we have*

$$\mathcal{I}(\Delta)^*B = E(\Delta)T^*(B) = T^*(B)E(\Delta) = E(\Delta)T^*(B)E(\Delta). \quad (10)$$

All the  $E$ -compatible instruments are determined as follows.

**Theorem 6** *Let  $E$  be an  $\Lambda$ -valued observable of  $\mathcal{H}$ . The relation*

$$\mathcal{I}(\Delta)\rho = T[E(\Delta)\rho] \quad (11)$$

*for all  $\Delta \in \mathcal{B}(\Lambda)$  and  $\rho \in \tau c(\mathcal{H})$  sets up a one-to-one correspondence between the  $E$ -compatible instruments  $\mathcal{I}$  and the  $E$ -compatible trace preserving operations  $T$ .*

From the above theorem, in every statistical measurement theory we have the following: *For any apparatus  $\mathbf{A}(\mathbf{x})$  measuring a  $\Lambda$ -valued observable  $E$ , there is an  $E$ -compatible trace preserving operation  $T$  such that the statistical property of  $\mathbf{A}(\mathbf{x})$  is represented as follows.*

$$\text{output distribution:} \quad \Pr\{\mathbf{x} \in \Delta | \rho\} = \text{Tr}[E(\Delta)\rho] \quad (12)$$

$$\text{collective output state:} \quad \rho_{\{\mathbf{x} \in \Delta\}} = \frac{T[E(\Delta)\rho]}{\text{Tr}[E(\Delta)\rho]} \quad (13)$$

## 6. MEASUREMENTS OF NONDEGENERATE OBSERVABLES

Let  $E$  be a  $\Lambda$ -valued observable for  $\mathcal{H}$ . We say that  $E$  is *nondegenerate* if the commutant of  $E$  is abelian. Two Borel families  $\{\varrho_x\}_{x \in \Lambda}$  and  $\{\varrho'_x\}_{x \in \Lambda}$  of density operators are said to be *E-equivalent* if they differ only on an  $E$ -null set, i.e.,

$$E\{x \in \Lambda \mid \varrho_x \neq \varrho'_x\} = 0.$$

**Theorem 7** *Let  $E$  be a nondegenerate  $\Lambda$ -valued observable of  $\mathcal{H}$ . The Bochner integral formula*

$$T\rho = \int_{\Lambda} \varrho_x \text{Tr}[\rho dE(x)] \quad (14)$$

*for all  $\rho \in \tau c(\mathcal{H})$  sets up a one-to-one correspondence between the  $E$ -compatible trace preserving operations  $T$  and the  $E$ -equivalence classes of the Borel families  $\{\varrho_x\}_{x \in \Lambda}$  of density operators indexed by  $\Lambda$ .*

From the above theorem, in any statistical measurement theory we conclude the following: *For any apparatus  $\mathbf{A}(\mathbf{x})$  measuring an  $\Lambda$ -valued observable  $E$ , there is a Borel family  $\{\varrho_x\}_{x \in \Lambda}$  of density operators uniquely up to  $E$ -equivalence such that the statistical property of  $\mathbf{A}(\mathbf{x})$  is represented as follows.*

$$\text{output distribution:} \quad \Pr\{\mathbf{x} \in \Delta | \rho\} = \text{Tr}[E(\Delta)\rho] \quad (15)$$

$$\text{output state:} \quad \rho_{\{\mathbf{x}=x\}} = \varrho_x \quad (16)$$

## 7. INDIRECT MEASUREMENT MODELS

An *indirect measurement model* for  $(\Lambda, \mathcal{H})$  is defined to be a 4-tuple  $(\mathcal{K}, \sigma, U, E)$  consisting of a separable Hilbert space  $\mathcal{K}$ , a density operator  $\sigma$  on  $\mathcal{K}$ , a unitary operator  $U$  on  $\mathcal{H} \otimes \mathcal{K}$ , and a  $\Lambda$ -valued observable  $E$  of  $\mathcal{K}$ .

If the apparatus  $\mathbf{A}(\mathbf{x})$  has the indirect measurement model  $(\mathcal{K}, \sigma, U, E)$ , then  $\mathbf{A}(\mathbf{x})$  has the instrument  $\mathcal{I}_{\mathbf{x}}$  defined by

$$\mathcal{I}_{\mathbf{x}}(\Delta)\rho = \text{Tr}_{\mathcal{K}}[(I \otimes E(\Delta))U(\rho \otimes \sigma)U^\dagger], \quad (17)$$

where  $\Delta \in \mathcal{B}(\Lambda)$  and  $\rho \in \tau c(\mathcal{H})$  [10], so that the statistical property of  $\mathbf{A}(\mathbf{x})$  is described by  $\mathcal{I}_{\mathbf{x}}$  with relations (DL1) and (DL2). The above instrument  $\mathcal{I}_{\mathbf{x}}$  is called the *instrument of  $\mathbf{A}(\mathbf{x})$* .

Now, we consider the following hypothesis.

**Indirect measurability hypothesis:** *For any indirect measurement model  $(\mathcal{K}, \sigma, U, E)$ , there is an apparatus  $\mathbf{A}(\mathbf{x})$  with the instrument  $\mathcal{I}_{\mathbf{x}}$  defined by (17).*

In general, an instrument  $\mathcal{I}_{\mathbf{x}}$  is said to be *realized* by an indirect measurement model  $(\mathcal{K}, \sigma, U, E)$  if (17) holds for any  $\rho \in \tau c(\mathcal{H})$ . In this case, the instrument  $\mathcal{I}$  is called *unitarily realizable*. Under the indirect measurability hypothesis, every unitarily realizable instrument represents the statistical property of an apparatus.

In the sequel, a statistical measurement theory  $(\mathcal{A}, \mathbf{M})$  is called a *standard measurement theory*, if it satisfies the indirect measurability hypothesis. It is natural to consider that any standard measurement theory is consistent with the standard formulation of quantum mechanics.

## 8. COMPLETE POSITIVITY

Let  $\mathcal{D} = \tau c(\mathcal{H})$  or  $\mathcal{D} = \mathcal{L}(\mathcal{H})$ . A linear transformation  $L$  on  $\mathcal{D}$  is called *completely positive (CP)* iff for any finite sequences of operators  $A_1, \dots, A_n \in \mathcal{D}$  and vectors  $\xi_1, \dots, \xi_n \in \mathcal{H}$  we have

$$\sum_{ij} \langle \xi_i | L(A_i^\dagger A_j) | \xi_j \rangle \geq 0.$$

The above condition is equivalent to that  $L \otimes I$  maps positive operators in the algebraic tensor product  $\mathcal{D} \otimes \mathcal{L}(\mathcal{K})$  to positive operators in  $\mathcal{D} \otimes \mathcal{L}(\mathcal{K})$  for any Hilbert space  $\mathcal{K}$ . Obviously, every CP superoperators are positive. A superoperator is CP if and only if its dual superoperator is CP. An instrument  $\mathcal{I}$  is called *completely positive (CP)* if every operation  $\mathcal{I}(\Delta)$  is CP. It can be seen easily from (17) that unitarily realizable instruments are CP. Conversely, the following theorem, proved in [5,14], asserts that every CP instrument is unitarily realizable.

**Theorem 8** *For any CP instrument  $\mathcal{I}$  for  $(\Lambda, \mathcal{H})$ , there is a separable Hilbert space  $\mathcal{K}$ , a unit vector  $\Phi$  in  $\mathcal{K}$ , a unitary operator  $U$  on  $\mathcal{H} \otimes \mathcal{K}$ , and a  $\Lambda$ -valued observable  $E$  of  $\mathcal{K}$  satisfying the relation*

$$\mathcal{I}(\Delta)\rho = \text{Tr}_{\mathcal{K}}[(I \otimes E(\Delta))U(\rho \otimes |\Phi\rangle\langle\Phi|)U^\dagger]$$

*for all  $\Delta \in \mathcal{B}(\Lambda)$  and  $\rho \in \tau_c(\mathcal{H})$ .*

The following theorem shows that the complete positivity of observable measuring instruments is determined by their total operations.

**Theorem 9** *Let  $E$  be a  $\Lambda$ -valued observable. Then, an  $E$ -compatible instrument is CP if and only if its total operation is CP.*

From the above theorem, in any standard measurement theory we conclude the following statement [5]: *The statistical equivalence classes of apparatuses  $\mathbf{A}(\mathbf{x})$  measuring a  $\Lambda$ -valued observable  $E$  with indirect measurement models are in one-to-one correspondence with the  $E$ -compatible trace preserving CP operations, where the statistical property is represented by (12) and (13).*

For the case of nondegenerate observables, we have the following simple characterizations.

**Theorem 10** *Let  $E$  be a nondegenerate  $\Lambda$ -valued observable. Then, every  $E$ -compatible operation is completely positive. Every  $E$ -compatible instrument is completely positive.*

From the above theorem and Theorem 8, in the statistical measurement theory we conclude: *Every apparatus measuring a nondegenerate ( $\Lambda$ -valued) observable is statistically equivalent to the one having an indirect measurement model.*

Every Borel family  $\{\varrho_x\}$  of density operators indexed by  $\Lambda$  defines an  $E$ -compatible trace preserving operation by Theorem 7, and it is automatically completely positive so that it is realized by an indirect measurement model. Thus, we conclude the following: *The statistical equivalence classes of apparatuses  $\mathbf{A}(\mathbf{x})$  measuring a nondegenerate  $\Lambda$ -valued observable  $E$  are in one-to-one correspondence with the Borel family  $\{\varrho_x\}$  of density operators indexed by  $\Lambda$ , where the statistical property is represented by (15) and (16).*

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